The Calibration Conundrum: Optimizers and (Joint) Objective Functions

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MathWorks Finance Conference 2025

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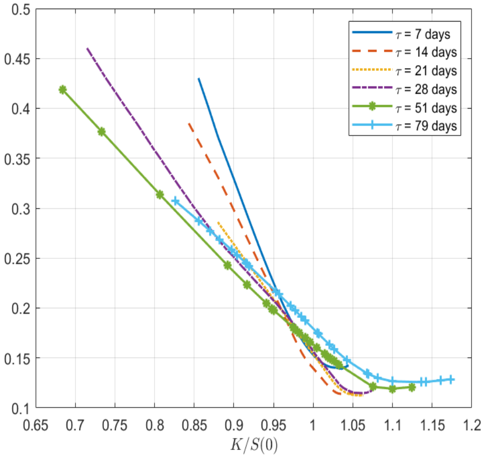
Introduction

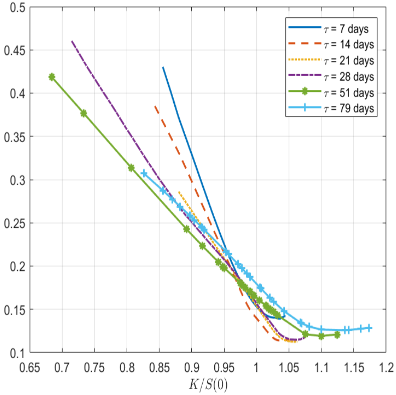
- Pricing of financial (structured) derivatives products
  - Trusted model(s)
  - Meaningful and consistent set of parameters
- Calibration
  - Market Data
  - Objective function: metrics of interest
  - Constraints: parameter space
  - Optimizer
- ‘Optimum’

Market implied volatilities (@ 3/05/2023)



S&P500

Market implied volatilities (@ 3/05/2023)



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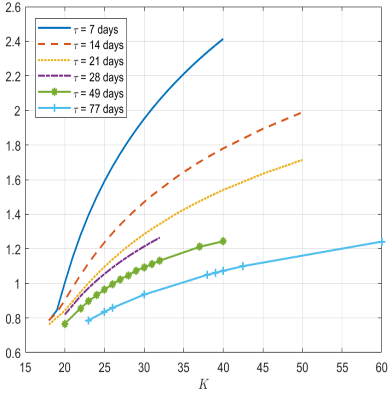
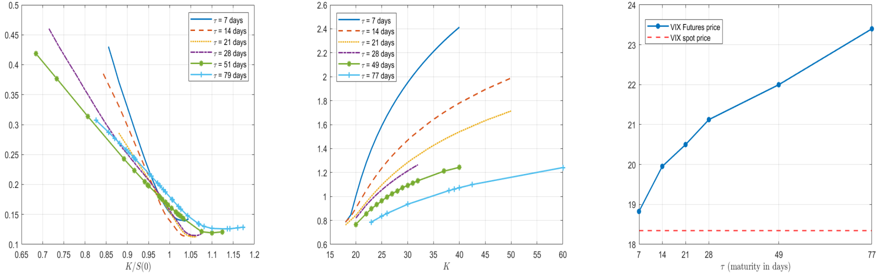


VIX

Market implied volatilities (@ 3/05/2023)



S&P500 VIX VIX Futures

Calibration

- Data: market implied volatilities of Out-of-the-Money (OTM) options
- Model with parameter set ϑ
  - $\pi^{mod}(\vartheta)$: model price of the option
- Calibration problem

$$\min_{\vartheta} F(\vartheta)$$

s.t. ϑ within the parameter limits of the chosen model

- Objective function

$$F(\vartheta) = \sum_{j=1}^{N} f_j(\vartheta)$$

N: total number of contracts in the dataset

- $f(\cdot)$: chosen metrics

Objective Function

- Metric: Mean Squared Error - monotone transformations (RMSE)
  - No arbitrage principle
- Argument
  - Implied volatilities extracted from the model $\rightarrow$ inversion Black-Scholes formula (efficient?)

$$f(\vartheta) = \left(iVol^{mod}(\vartheta) - iVol^{mkt}\right)^2,$$

  - Rescaled prices (approximation first order)

$$f(\vartheta) = \left(\frac{\pi^{mod}(\vartheta) - \pi^{mkt}}{Vega^{mkt}}\right)^2,$$

Fourier transform for derivatives valuation

- Pricing according to Eberlein, Glau, Papapantoleon (2010) method
- Underlying $S = (S_t)_{0\leq t\leq T}$ with risk driver $X = (X_t)_{0\leq t\leq T}$
- Derivative on S with payoff $g(X_T)$, with price

$$\pi^{mod}(\vartheta) = e^{-rT}\frac{e^{-Rs}}{\pi}\int_0^\infty \Re\left(e^{-\mathrm{i}us}\,\underbrace{\phi_{X_T}(u-\mathrm{i}R;\vartheta)}\,\underbrace{\hat{g}(\mathrm{i}R-u)}\right)du,$$

- ϕ_{X_T}: characteristic function of X_T; $s = -\ln S_0$
- $\hat{g}$: Fourier transform of g
- R: dampening parameter
- Integrals: `integral` Matlab function, fully vectorized for speed - `'ArrayValued'`
- Inverse Fourier Transform: numerical recovery of density functions from characteristic function

A Toy Example (I)

- Methodology
  - Choose a model and fix the parameters
  - Price options and extract corresponding implied volatilities
  - Calibrate the model and check if original parameters can be recovered

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- Model: CGMY (Carr et al., 2002), parameter set $\vartheta = \{C, G, M, Y\}$

- Characteristic function $\phi(u) = exp(\varphi(u)t)$, for

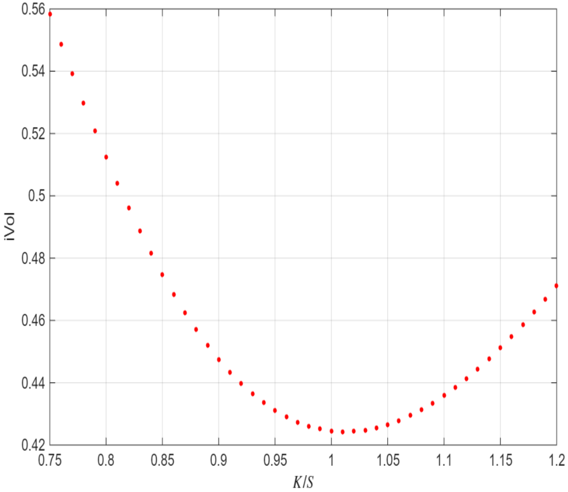
$$\varphi(u) = C\Gamma(-Y)\left((G + \mathrm{i}u)^Y - G^Y + (M - \mathrm{i}u)^Y - M^Y\right),$$

- Fourier transform option payoff

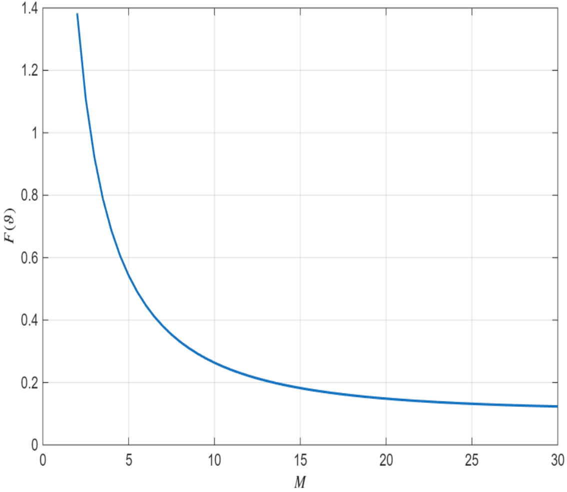
$$\hat{g}(\mathrm{i}R - u) = \frac{K^{1-R-\mathrm{i}u}}{(-R - \mathrm{i}u)(1 - R - \mathrm{i}u)}$$

- $R > 1$ for call options, $R < 0$ for put options

A Toy Example (II): Pseudo implied volatilities



A Toy Example (III): Objective Function

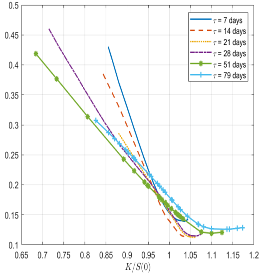


A Toy Example (IV): 'Algorithm' risk

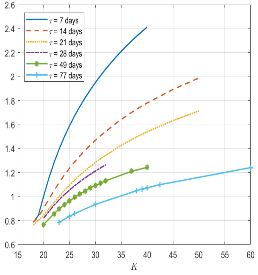
$$F(\vartheta) = \sum_{j=1}^{N} \left(iVol^{mod}\left(K_j;\vartheta\right) - iVol^{mkt}\left(K_j\right) \right)^2,$$

	C	G	M	Y	O.F.	F. Evals.	RMSE iVol	APE Price	CPU (sec)
Original parameters	0.1	2	3.5	1.5	-	-	-	-	-
`lsqnonlin`	0.118949	2.295202	3.820629	1.458416	4.1465E-06	270	3.0023E-04	1.5474E-03	3.46
`fmincon`	0.100203	2.003428	3.503760	1.499521	5.8969E-10	970	3.5804E-06	1.8291E-05	5.97
`ga`	0.298868	3.945294	5.585775	1.220572	1.3802E-04	18853	3.5804E-06	9.0510E-03	270.86
`ga` hybrid (`fmincon`)	0.100297	2.004999	3.505465	1.499301	1.2518E-09	19353	5.2166E-06	2.6913E-05	265.09
`gs`	0.100161	2.002706	3.502956	1.499622	3.6686E-10	66871	2.8240E-06	1.4589E-05	281.65
`ms` (`fmincon`)	0.100154	2.002600	3.502847	1.499636	3.3900E-10	58526	2.7147E-06	1.3949E-05	394.12
`ms` (`lsqnonlin`)	0.108676	2.140973	3.653332	1.480156	1.0818E-06	15755	1.5335E-04	7.3013E-04	171.65

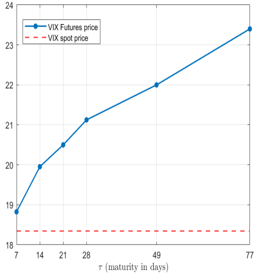
A more sophisticated problem



S&P500



VIX



VIX Futures

The financial model

- Equity index: $S(t) = S(0)e^{rt+X(t)}$, $S(0)$: spot price, r: risk free rate of interest
- Log-returns process (equity index): $X(t) = L(T(t))$

$$X(t) = -\varphi_{J_1}(-\mathrm{i}\sigma_1)T(t) + \sigma_1\underbrace{(J_{1,-}(T(t)) + J_{1,+}(T(t)))}_{J_1(T(t))}$$

- $T(t) = \int_0^t v(s)ds$: time change/business clock/integrated variance
- Activity rate process: $v(t) = v(0) + \kappa\theta t + Y(T(t))$

$$dv(t) = \kappa(\theta - v(t_-))\,dt - \eta_1 dJ_{1,-}(T(t)) + \eta_2 dJ_2(T(t))$$

  - $J_1(t)$: pure jump Lévy process - CGMY
  - $J_{1,-}(t)$: process with the negative (compensated) jumps of J_1
  - $J_{1,+}(t)$: process with the positive (compensated) jumps of J_1
  - $J_2(t)$: independent, positive, pure jump Lévy process - Gamma process

Log-returns characteristic function

$$\phi_X(u;t) = \mathbb{E}\left(e^{\mathrm{i}uX(t)}\right) = e^{D_0(u;t)+D_1(u;t)v(0)}$$

with D_0, D_1 solutions to the system

$$\begin{aligned}
D_0'(u;t) &= \kappa\theta D_1(u;t), \\
D_0(u;0) &= 0 \\
D_1'(u;t) &= \varphi_L(u) - \kappa D_1(u;t) + \varphi_{J_{1,-}}\left(\mathrm{i}\eta_1 D_1(u;t) + \sigma_1 u\right) \\
&\quad -\varphi_{J_{1,-}}\left(\sigma_1 u\right) + \varphi_{J_2}\left(-\mathrm{i}\eta_2 D_1(u;t)\right), \\
D_1(u;0) &= 0
\end{aligned}$$

- 'Numerics': `ode45` (with vector 'parameter' u)

The ‘fear’ index

- VIX index (CBOE)

$$\bar{V}(0,\Delta_\tau) = 100 \times \sqrt{\frac{2}{\Delta_\tau} e^{r\Delta_\tau} \sum_i \frac{\Delta K_i}{K_i^2} O(K_i) - \frac{1}{\Delta_\tau}\left(\frac{F_S(0,\Delta_\tau)}{K_0} - 1\right)^2}$$

$$= 100 \times \sqrt{-\frac{2}{\Delta_\tau}\mathbb{E}\left(\ln \frac{S(\Delta_\tau)}{F_S(0,\Delta_\tau)}\right)}$$

$$= 100 \times \sqrt{\frac{1}{\Delta_\tau}\left(\mathbb{V}ar(L(1)) + \frac{1}{3}c_3(L(1)) + \frac{1}{12}c_4(L(1)) + \cdots\right)\mathbb{E}\left(T(\Delta_\tau)\right)}$$

- Variance and higher order cumulants of log-returns base process (skewness & excess kurtosis)
- $V(t, t+\Delta_\tau)^2 = -\frac{2}{\Delta_\tau}\left(-\varphi_{J_1}(-\mathrm{i}\sigma_1) + \sigma_1\mathbb{E}(J_1(1))\right)\left(c_0(\Delta_\tau) + c_1(\Delta_\tau)v(t)\right)$

  for $c_0(\Delta_\tau), c_1(\Delta_\tau)$ suitably defined (explicitly known)

VIX squared characteristic function

$$\phi_{V^2}(h;t) = \mathbb{E}\left(e^{\mathrm{i}hV(t,t+\Delta_\tau)^2}\right) = e^{A(h;\Delta_\tau)+A_0(h;t)+A_1(h;t)v(0)}$$

with

$$A(h;\Delta_\tau) = -\mathrm{i}h\frac{2b_2}{\Delta_\tau}c_0(\Delta_\tau),$$

and A_0, A_1 solutions to the system

$$\begin{aligned}
A_0'(h;t) &= \kappa\theta A_1(h;t), \\
A_0(h;0) &= 0 \\
A_1'(h;t) &= -\kappa A_1(h;t) + \varphi_{J_{1,-}}\left(i\eta_1 A_1(h;t)\right) + \varphi_{J_2}\left(-i\eta_2 A_1(h;t)\right), \\
A_1(h;0) &= -\mathrm{i}h\frac{2b_2}{\Delta_\tau}c_1(\Delta_\tau)
\end{aligned}$$

- 'Numerics': `ode45` (with vector initial conditions)

Fourier based derivatives pricing

- Option pricing according to Eberlein, Glau and Papapantoleon (2010)

- S&P option price with strike K and maturity τ

$$e^{-r\tau}\frac{e^{-Rs}}{\pi}\int_0^{\infty}\Re\left(e^{-ius}\phi_X(u-iR;\tau)\frac{K^{1-R-\mathrm{i}u}}{(1-R-\mathrm{i}u)(-R-\mathrm{i}u)}\right)du$$

- VIX call option price with strike K and maturity τ

$$100\times\frac{e^{-r\tau}}{\sqrt{\pi}}\int_0^{\infty}\Re\left(\phi_{V^2}(u-\mathrm{i}R;\tau)\frac{(1-\mathrm{erf}(K\sqrt{R+\mathrm{i}u}))}{2(R+\mathrm{i}u)^{3/2}}\right)du$$

erf$(\cdot)$ denoting the error function of a complex argument (Paul Godfrey - File Exchange MATLAB Central)

- VIX futures price for maturity τ

$$F_V(0,\tau)=100\times\frac{1}{2\sqrt{\pi}}\int_0^{\infty}\frac{1-\phi_{V^2}(\mathrm{i}u;\tau)}{u^{3/2}}du$$

Joint Calibration

- Calibration problem

$$\min_{\vartheta} F(\vartheta)$$

s.t. ϑ within the parameter limits of the chosen model

- Objective function

$$F(\vartheta) = \sum_{\lambda=1}^{2} \frac{1}{N_\lambda} \sum_{j=1}^{N_\lambda} f_\lambda(j;\vartheta) + \frac{1}{N_F} \sum_{l=1}^{N_F} f(l;\vartheta)$$

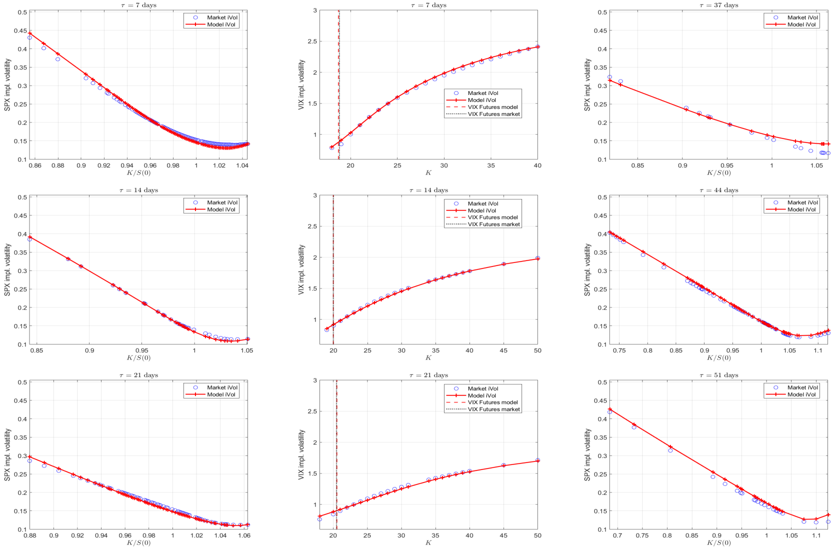
- Option prices relative error (approximation first order)

$$f_\lambda(j;\vartheta) = \left(\frac{\pi_\lambda^{mod}(j;\vartheta) - \pi_\lambda^{eod}(j)}{\mathrm{iVol}_\lambda^{eod}(j)\, Vega_\lambda(j)} \right)^2, \quad \lambda = 1,2,\ j = 1,\dots,N_\lambda,$$

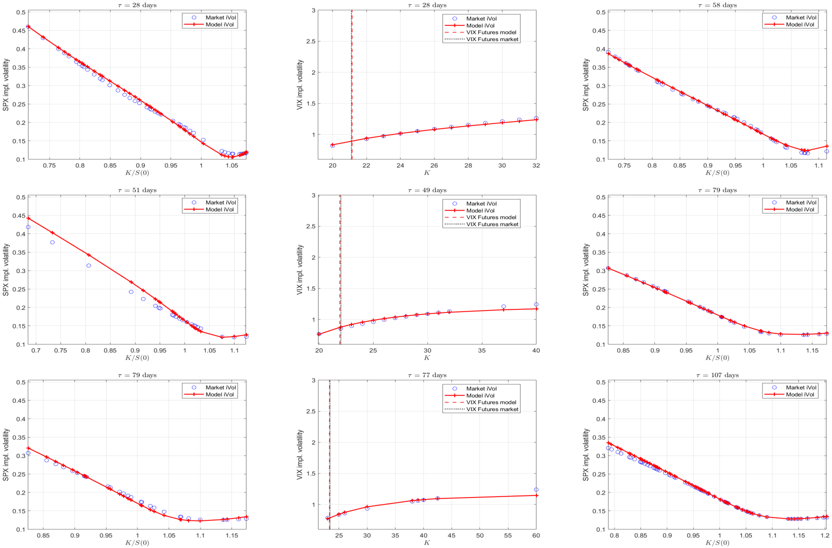
- Futures prices relative error

$$f(l;\vartheta) = \left(\frac{\pi^{mod}(l;\vartheta) - \pi^{eod}(l)}{\pi^{eod}(l)} \right)^2, l = 1,\dots,N_F$$

Piece-wise calibration (I)



Piece-wise calibration (II)



Conclusions

- Model Calibration
  - Pricing of structured products
  - Hedging
- Objective Function specification
  - Metrics
  - Magnitude of quantities of relevance
- Local optima
- 'Algorithm' risk
- Simpler solutions sometimes are the better

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